PMM U.S.S.R.,Vol.45,pp. 755-760

0021-8928/82/6 0755 $7.50/0
UDC 62-50

MIXED STRATEGIES AND THEIR APPLICATION IN THE ENCOUNTER-EVASION DIFFERENTIAL GAMES*

S.A. VAKHRAMEEV

A positional encounter-evasion differential game with geometric constraints on the players' controls, depending on the system's state, is examined. The concept of the players' mixed strategies is introduced and an alternative is proved which asserts that either the positional encounter game or the positional evasion game is always solvable. The paper continues the investigations in /1-4/.

1. Dynamic system. Let the controlled system's behavior be described by the equation

$$x^{\cdot} = f(t, x, u, v),\ x \in R^n,\ u \in R^p,\ v \in R^q \tag{1.1}$$

where x is the system's phase coordinate vector, u and v are the controls of the first and second players, respectively. By Ω^m we denote the space of all nonempty compacta in R^m with the Hausdorff metric h. We assume as specified the mappings

$$P : R \times R^n \to \Omega^p,\ Q : R \times R^n \to \Omega^q \tag{1.2}$$

satisfying the following conditions:

1°. The mappings $t \mapsto P(t, x), t \mapsto Q(t, x)$ are measurable, while the mappings $x \mapsto P(t, x)$, $x \mapsto Q(t, x)$ satisfy the Lipschitz conditions

$$h(P(t, x), P(t, y)) \leqslant \alpha(t)|x-y|,\ h(Q(t, x), Q(t, y)) \leqslant \alpha(t)|x-y| \tag{1.3}$$

2°. Measurable mappings

$$P_0 : R \to \Omega^p,\ Q_0 : R \to \Omega^q$$

exist such that

$$\bigcup_{x \in R^n} P(t,x) \subset P_0(t),\quad \bigcup_{x \in R^n} Q(t,x) \subset Q_0(t)$$

The following assumptions are made concerning the function f in the right-hand side of (1.1).

1°. The function $f(\cdot, x, u, v) : R \to R^n$ is measurable.

2°. For all $x \in R^n, u \in P_0(t), v \in Q_0(t)$

$$|f(t, x, u, v)| \leqslant k(t)(1 + |x|) \tag{1.4}$$

3°. For all $x, y \in R^n, u', u'' \in P_0(t), v', v'' \in Q_0(t)$

$$|f(t, x, u', v') - f(t, y, u'', v'')| \leqslant \beta(t)(|x-y| + |u'-u''| + |v'-v''|) \tag{1.5}$$

Here the functions $\alpha, \beta, k : R \to R$ are nonnegative and locally summable.

2. On families of Radon measures on R^m. Let μ_t, $t \in R$, be a family of Radon measures on R^m, depending on parameter $t \in R$. We say that the family μ_t, $t \in R$, is weakly measurable if for any continuous function $\varphi : R^m \to R$ with compact support the function

$$t \mapsto \langle \mu_t, \varphi(u)\rangle = \int_{R^m} \varphi(u)\, d\mu_t$$

is (Lebesgue) measurable. Of particular interest are weakly measurable families μ_t, $t \in R$, of Radon measures on R^m, whose supports $\operatorname{supp} \mu_t$ satisfy the inclusion

$$\operatorname{supp} \mu_t \subset F(t) \in \Omega^m$$

for almost all $t \in R$. About such families we shall say that they are concentrated on the

---

*Prikl.Matem.Mekhan.,45,No.6,1001-1008,1981 mapping $F: R \to \Omega^m$. If the mapping $F: R \to \Omega^m$ is measurable, then such families exist. As a matter of fact, let $f: R \to R^m$ be an arbitrary measurable branch of mapping $F: R \to \Omega^m$. Then the family of Dirac measures $\delta_{f(t)}, t \in R$ yields an example of a weakly measurable family of Radon probability measures, concentrated on mapping F.

Proposition 1. Let μ_t, $t \in R$, be a weakly measurable family of Radon measures on R^m, concentrated on a measurable mapping $F: R \to \Omega^m$. Then for any function $f: R \times R^m \to R$ measurable in $t \in R$ and continuous in $u \in R^m$, the function

$$t \mapsto \langle \mu_t, f(t, u)\rangle = \int_{R^m} f(t, u)\, d\mu_t$$

is (Lebesgue) measurable.

The proof is based on the theorems of Luzin and of Scorza—Dragoni (see /5,6/).

For any compactum $A \in \Omega^m$ we denote the set of all Radon probability measures concentrated on A by A_c. It is well known that the set A_c is weakly (sequentially) compact /7/. Furthermore, it can be proved that the equality

$$\operatorname{conv} \varphi(A) = \{\langle \mu, \varphi(u)\rangle \mid \mu \in A_c\} \tag{2.1}$$

holds for any continuous function $\varphi: R^m \to R^n$. As a matter of fact, let $c(U, \psi)$ be the support function of $U \in \Omega^n$

$$c(U, \psi) = \max_{u \in U} (u, \psi)$$

The set $D = \{\langle \mu, \varphi(u)\rangle \mid \mu \in A_c\}$ is convex and compact and its support function is

$$c(D, \psi) = \max_{\mu \in A_c} (\psi, \langle \mu, \varphi(u)\rangle) \leqslant c(\varphi(A), \psi)$$

Therefore, $D \subset \operatorname{conv} \varphi(A)$. However, we obtain equality (2.1) because

$$\operatorname{conv} D = D \supset \varphi(A)$$

Proposition 2.2. Let μ_t, $t \in R$, and ν_t, $t \in R$ be weakly measurable families of Radon measures on R^m and R^n, concentrated on measurable mappings

$$F: R \to \Omega^m,\ G: R \to \Omega^n$$

respectively. Then the family $\eta_t = \mu_t \otimes \nu_t$, $t \in R$, of Radon measures on R^{n+m} is weakly measurable and is concentrated on the mesurable mapping

$$t \mapsto F(t) \times G(t): R \to \Omega^{n+m}$$

The proof follows from Proposition 2.1.

3. Strategies and motions.

For an arbitrary mapping

$$F: R \times R^n \to \Omega^m$$

measurable in $t \in R$, by $F_c(x; t_*, t^*)$ we denote the collection of all weakly measurable families $\mu_t, t \in R$, of Radon probability measures on R^m, concentrated on mappings $F(\cdot, x): R \to \Omega^m$ for $t_* \leqslant t < t^*$. A mapping which associates a nonempty subset of $P_c(x; t, \infty)$ $(Q_c(x; t, \infty))$ with an arbitrary position (t, x) is called a mixed strategy U_c (V_c) of the first (second) player.

Suppose that the first player chose a mixed strategy U_c. We consider a partitioning Δ of the semiaxis $[t_0, \infty)$ into a system of half-open intervals of the form

$$\tau_i \leqslant t < \tau_{i+1},\ i = 0, 1, \ldots, \tau_0 = t_0,\ \tau_i \to \infty\ (i \to \infty)$$

Let $|\Delta| = \sup_i (\tau_{i+1} - \tau_i)$. We look at the differential equation

$$\begin{gathered} x_\Delta^{\cdot}(t) = \langle \mu_t^{(i)} \otimes \nu_t^{(i)}, f(t, x_\Delta(t), u, v)\rangle \\ \tau_i \leqslant t < \tau_{i+1},\ i = 0, 1, \ldots, x_\Delta(t_0) = x_0 \\ \mu_t^{(i)} \in U_c(\tau_i; x_\Delta(\tau_i)),\ \nu_t^{(i)} \in Q_c(x_\Delta(\tau_i); \tau_i, \tau_{i+1}) \end{gathered} \tag{3.1}$$

We see that it has the solution $x_\Delta(t) = x_\Delta(t; t_0, x_0, U_c, \nu_t)$ continuable onto the half-line $t \geqslant t_0$. Indeed, since $\mu_t^{(i)}$ and $\nu_t^{(i)}$ are probability measures for almost all t,

$$|\langle \mu_t^{(i)} \otimes \nu_t^{(i)}, f(t, x, u, v)\rangle| \leqslant |f(t, x, u, v)| \leqslant k(t)(1 + |x|),$$

$$|\langle \mu_t^{(i)} \otimes \nu_t^{(i)}), f(t, y, u, v)\rangle - f(t, x, u, v)\rangle| \leqslant |f(t, y, u, v) - f(t, x, u, v)| \leqslant \beta(t)|x - y|$$

According to Propositions 2.1 and 2.2 the functions $t \mapsto \langle \mu_t^{(i)} \otimes \nu_t^{(i)}, f(t, x, u, v)\rangle$ are measurable. Consequently, the existence of a unique (under a specific choice of families $\mu_t^{(i)}$ and $\nu_t^{(i)}, i = 0, 1, \ldots$) solution, continuable onto the half-line $t_0 \leqslant t < \infty$, of Eq.(3.1) follows from well-known results in the theory of differential equations /5/. The solution $x_\Delta(t) = x_\Delta(t; t_0, x_0, U_c, \nu_t)$ of

Eq. (3.1) is called the Euler polygonal line generated by the first player's mixed strategy U_c. The Euler polygonal line generated by the second player's mixed strategy V_c is defined analogously.

Proposition 3.1. Every Euler polygonal line of the first or second player is a solution of the differential inclusion

$$x^{\cdot} \in \operatorname{conv} f(t, x, P_0(t), Q_0(t)) \tag{3.2}$$

The proof is obtained from the results of Sect. 2. From Proposition 3.1 follows the correctness of the next definition.

Every function $x(t),\ t \geqslant t_0$, for which we can find, on any finite interval $t_0 \leqslant t \leqslant t_1$, a sequence $\{x_{\Delta_k}\}$ of Euler polygonal lines

$$x_{\Delta_k}(t) = x_{\Delta_k}(t; t_0, x_0^{(k)}, U_c, \nu_t^{(k)}) \quad (x_{\Delta_k}(t) = x_{\Delta_k}(t; t_0, x_0^{(k)}, V_c, \mu_t^{(k)}))$$

generated by the first (second) player's strategy U_c (V_c), such that

$$x_{\Delta_k}(t) \rightrightarrows x(t),\ t_0 \leqslant t \leqslant t_1,\ x_0^{(k)} \to x_0,\ |\Delta_k| \to 0,\ k \to \infty$$

is called a motion $x(t) = x(t; t_0, x_0, U_c)$ $(x(t) = x(t; t_0, x_0, V_c))$ generated by the first (second) player's mixed strategy U_c (V_c). It can be proved that every motion of the first or second player, starting from point x_0 at instant $t = t_0$ is a solution of the differential inclusion

$$x^{\cdot} \in \operatorname{conv} f(t, x, P(t, x), Q(t, x)),\ x(t_0) = x_0$$

4. Encounter-evasion differential game.

The game being examined is made up of the following two problems. Let nonempty closed sets M and N in the position space $R \times R^n$, an initial position (t_0, x_0) and an instant $\vartheta \geqslant t_0$ be specified.

Problem 1. Find the first player's mixed strategy $U_c{}^*$ ensuring the contact

$$(t, x(t)) \in N,\ t_0 \leqslant t < \tau,\ (\tau, x(\tau)) \in M,\ \tau = \tau(x(\cdot)) \leqslant \vartheta$$

for all motions $x(t) = x(t; t_0, x_0, U_c{}^*)$.

Problem 2. Indicate open neighborhoods $G(M)$ and $H(N)$ of sets M and N, as well as a second player's mixed strategy $V_c{}^*$, such that the contact

$$(t, x(t)) \in H(N), t_0 \leqslant t < \tau, (\tau, x(\tau)) \in G(M), \tau = \tau(x(\cdot)) \leqslant \vartheta$$

is exluded for all motions $x(t) = x(t; t_0, x_0, V_c{}^*)$.

Problem 1 is called the problem of encounter with set M inside set N by the instant ϑ, while Problem 2 is called the problem of evading $G(M)$ inside $H(N)$ up to the instant ϑ.

5. Stable sets.

We say that a set $W \subset R \times R^n$ is u_c-stable (v_c stable) if for any $(t_*, x_*) \in W,\ t^* > t_*$ and $\nu_t{}^* \in Q_c(x_*; t_*, t^*)$ $(\mu_t{}^* \in P_c(x_*; t_*, t^*))$ there exists $\mu_t{}^* \in P_c(x_*; t_*, t^*)$ $(\nu_t{}^* \in Q_c(x_*; t_*, t^*))$ such that the solution $x(t),\ t_* \leqslant t \leqslant t^*$ of the differential equation

$$x^{\cdot} = \langle \mu_t{}^* \otimes \nu_t{}^*,\ f(t, x, u, v)\rangle,\quad x(t_*) = x_* \tag{5.1}$$

satisfies the condition $(t^*, x(t^*)) \in W$ or the condition $(\tau, x(\tau)) \in M$ $((\tau, x(\tau)) \notin H(N))$ for some $\tau,\ t_* \leqslant \tau \leqslant t^*$.

Theorem 5.1. If set $W \subset R \times R^n$ is u_c-stable (v_c-stable), then its closure $\bar{W} = \operatorname{Cl} W$ is u_c-stable (v_c-stable).

We carry out the proof, say, for a u_c-stable set W. The following proposition is valid.

Proposition 5.1. Let $\nu_t{}^* \in Q_c(x_*; t_*, t^*)$ and $x_*{}^{(k)} \to x_*$ as $k \to \infty$ Then there exist $\nu_t{}^{(k)} \in Q_c(x_*{}^{(k)}; t_*, t^*)$ such that $\nu_t{}^{(k)} \to \nu_t{}^*$ weakly for almost all $t,\ t_* \leqslant t \leqslant t^*$.

Indeed, let $O^{(i)} = \{O_1^{(i)}, \ldots, O_k^{(i)}, \ldots\}$ be a sequence of coverings of space R^q by open sets, such that

$$\operatorname{diam} O^{(i)} = \max_j \operatorname{diam} O_j^{(i)} \to 0 \quad (i \to \infty)$$

Let $\alpha_i^{(j)},\ i = 1, \ldots$ be a partitioning of unity, subordinate to covering $O^{(j)} = \{O_1^{(j)}, \ldots, O_i^{(j)}, \ldots\}$. We set

$$\lambda_i^{(j)}(t) = \langle \nu_t^{(j)},\quad \alpha_i^{(j)}(v)\rangle,\quad \nu_t^{*j} = \sum_{i=1}^{\infty} \delta_{v_i^{(j)}(t)} \lambda_i^{(j)}(t)_i \quad i, j = 1, 2, \ldots$$

where $v_i^{(j)}(t),\ t \in R$, are measurable functions such that $v_i^{(j)}(t) \in O_i^{(j)} \cap Q(t, x_*)$, if the latter set is nonempty, and $v_i^{(j)}(t) = v(t)$, otherwise, where $v(t), t \in R$, is an arbitrary measurable branch of mapping $t \mapsto Q(t, x_*)$. From the continuity of mapping $x \mapsto Q(t, x)$ it follows that we can find a sequence of measurable functions $v_i^{(j,k)}(t)$ such that $v_i^{(j,k)}(t) \in Q(t, x_*^{(k)})$ and $v_i^{(j,k)}(t) \to v_i^{(j)}(t)$ for almost all t, $t_* \leqslant t \leqslant t^*$ uniformly relative to $i, j = 1, 2, \ldots$. We set

$$\nu_t^{(k)} = \sum_{j=1}^{\infty} \lambda_j^{(k)}(t)\, \delta_{v_j^{(k,k)}}(t), \quad k = 1, 2, \ldots$$

From the very method of construction of the sequence $\{\nu_t^{(k)}\}$ it follows that $\nu_t^{(k)} \in Q_c(x_*^{(k)}; t_*, t^*)$ and $\nu_t^{(k)} \to \nu_t^* \ (k \to \infty)$ weakly for almost all $t, t_* \leqslant t \leqslant t^*$. Proposition 5.1 is proved.

Let us assume now that the assertion of Theorem 5.1 does not hold. Then there exist $(t_*, x_*) \in \overline{W}$, $t^* > t_*$, $\nu_t^* \in Q_c(x_*; t_*, t^*)$, such that for any measure $\mu_t^* \in P_c(x_*; t_*, t^*)$ the solution $x(t)$, $t_* \leqslant t \leqslant t^*$, of Eq.(5.1) satisfies the conditions

$$(t^*, x(t^*)) \notin \overline{W}, \ (\tau, x(\tau)) \notin M, \ t_* \leqslant \tau \leqslant t^*$$

However, since $(t_*, x_*) \in \overline{W}$, a sequence $\{(t_*^{(k)}, x_*^{(k)})\}$ of points of W exists such that $(t_*^{(k)}, x_*^{(k)}) \to (t_*, x_*)$, $k \to \infty$. By Proposition 5.1 a sequence $\nu_t^{(k)} \in Q_c(x_*^{(k)}, t_*, t^*)$ exists such that $\nu_t^{(k)} \to \nu_t^* \ (k \to \infty)$ weakly for almost all t, $t_* \leqslant t \leqslant t^*$. Finally, from the u_c-stability of set W follows the existence of $\mu_t^{(k)} \in P_c(x_*^{(k)}; t_*^{(k)}, t^*)$ such that the solutions $x_k(t)$, $t_*^{(k)} \leqslant t \leqslant t^*$, $k = 1, 2 \ldots$, of the differential equation

$$x_k^{\cdot}(t) = \langle \mu_t^{(k)} \otimes \nu_t^{(k)}, f(t, x_k(t), u, v)\rangle, \quad x(t_*^{(k)}) = x_*^{(k)}$$

satisfy either the condition $(t^*, x_k(t^*)) \in W$ or the condition $(\tau_k, x_k(\tau_k)) \in M$ for some τ_k, $t_*^{(k)} \leqslant \tau_k \leqslant t^*$. It can be shown that some subsequence of sequence $\{x_k(t)\}$ converges uniformly to an absolutely continuous function $x(t)$, $t_* \leqslant t \leqslant t^*$, which satisfies Eq.(5.1) with some $\mu_t^* \in P_c(x_*; t_*, t^*)$. This fact leads to a contradiction. The theorem is proved.

6. Derivation of the basic estimate. Let the functions $x(t)$, $y(t)$, $t \geqslant t_*$ satisfy Eqs.(6.1) and (6.2), respectively,

$$x^{\cdot} = \langle \mu^*_t \otimes \nu_t, f(t, x, u, v)\rangle, \ x(t_*) = x_* \tag{6.1}$$

$$y^{\cdot} = \langle \mu_t \otimes \nu_t^*, f(t, y, u, v)\rangle, \ y(t_*) = y_* \tag{6.2}$$

Here the families $\mu_t \in P_c(y_*; t_*, \infty)$ and $\nu_t \in Q_c(x_*; t_*, \infty)$ are arbitrary, while the families $\mu_t^* \in P_c(x_*; t_*, \infty)$ and $\nu^*_t \in Q_c(y_*; t_*, \infty)$ were chosen from the condtions

$$\max_{\nu \in Q_c(t, x_*)} K(t, x_*, z_*, \mu_t^*, \nu) = \min_{\mu \in P_c(t, x_*)} \max_{\nu \in Q_c(t, x_*)} K(t, x_*, z_*, \mu, \nu), t \geqslant t_*$$

$$\min_{\mu \in P_c(t, y_*)} K(t, y_*, z_*, \mu, \nu_t^*) = \max_{\nu \in Q_c(t, y_*)} \min_{\mu \in P_c(t, y_*)} K(t, y_*, z_*, \mu, \nu), t \geqslant t_*$$

$$K(t, x, z, \mu, \nu) = \langle \mu \otimes \nu, f(t, x, u, v)\rangle, \quad z_* = x_* - y_*$$

The possibility of making such a choice can be substantiated on the basis of the results in Sect.2 and of the Filippov—Kasten theorem (see /5/).

Theorem 6.1. For an arbitrary bounded domain $G \subset R \times R^n$ containing the positions (t_*, x_*) and (t_*, y_*) there exists a locally summable function $m_G: R \to R$ such that the inequality

$$\rho^2(t) \leqslant \rho^2(t_*)\Big(1 + 2\int_{t_*}^{t} \gamma(\tau)\, d\tau\Big) + \int_{t_*}^{t} \varphi(t_*, \tau)\, m(\tau)\, d\tau \tag{6.3}$$

$$m(t) = 4g\beta(t) + 8m_G(t), \quad g = \operatorname{diam} G, \quad \varphi(t_*, t) = \int_{t_*}^{t} m_G(\tau)\, d\tau$$

$$\gamma(t) = \beta(t) + 2\beta(t)\, \alpha(t), \ \rho(t) = |x(t) - y(t)|$$

is valid for all $t \geqslant t_*$.

The proof of Theorem 6.1 is carried out analogously to /1,2/ and is based on the following assertion which can be proved by using the standard construction of a partitioning of unity.

Proposition 6.1. Let the mapping $F: R^k \to \Omega^m$ satisfy the Lipschitz condition

$$h(F(x), F(y)) \leqslant L_F |x - y|$$

Then for any measure $\mu(x) \in F_c(x)$ there exists a measure $\mu(y) \in F_c(y)$ such that

$$|\langle\mu(x), \varphi(u)\rangle - \langle\mu(y), \varphi(u)\rangle| \leqslant L_\varphi L_F |x-y|$$

for any function $\varphi: R^m \to R$ satisfying the Lipschitz condition

$$|\varphi(u) - \varphi(v)| \leqslant L_\varphi |u - v|$$

7. **Extremal barrier.** Let $W \subset R \times R^n$ be a nonempty closed set in the position space $R \times R^n$. The first and second player's mixed strategies U_c^e and V_c^e extremal to this set are defined thus. Let $\Gamma_\tau = \{(t, x) \quad t = \tau\}$. If $\Gamma_{t_*} \cap W = \varnothing$, we set

$$U_c^e(t_*, x_*) = P_c(x_*; t_*, \infty), \quad V_c^e(t_*; x_*) = Q_c(x_*; t_*, \infty)$$

otherwise

$$U_c^e(t_*, x_*) = \{\mu_t^* \in P_c(x_*; t_*, \infty) | \max_{v \in Q_c(t, x_*)} K(t, x_*, x_* - w_*, \mu_t^*, v) = \min_{\mu \in P_c(t, x_*)} \max_{v \in Q_c(t, x_*)} K(t, x_*, x_* - w_*, \mu, v), t \geqslant t_*\}$$

$$V_c^e(t_*, x_*) = \{v_t^* \in Q_c(x_*; t_*, \infty) | \min_{\mu \in P_c(t, x_*)} K(t, x_*, w_* - x_*, \mu, v) = \max_{v \in Q_c(t, x_*)} \min_{\mu \in P_c(t, x_*)} K(t, x_*, w_* - x_*, \mu, v), t \geqslant t_*\}$$

where W_* is the vector of the section $W(t_*)$ of set W by the hyperplane Γ_{t_*}, lying closest to x_*. Using estimate (6.3), the next two assertions can be proved analogously to /1,2/.

Theorem 7.1. Let $W \subset R \times R^n$ be a closed u_c-stable set, U_c^e be a mixed strategy extremal to this set, and $(t_0, x_0) \in W$. Then for any motion $x(t) = x(t; t_0, x_0, U_c^e)$ the inclusion $(t, x(t)) \in W$ is fulfilled up to the instant τ that $(\tau, x(\tau)) \in M$.

Theorem 7.2. Let $W \subset R \times R^n$ be a closed v_c-stable set, V_c^e be a mixed strategy extremal to this set, and $(t_0, x_0) \in W$. Then for any motion $x(t) = x(t; t_0, x_0, V_c^e)$ the inclusion $(t, x(t)) \in W$ is fulfilled up to the instant τ that $(\tau, x(\tau)) \notin H(N)$.

8. **Alternative.** The following theorem is valid.

Theorem 8.1. Let the condition formulated in Sect.1 be fulfilled. Then either Problem 1 or Problem 2 is always solvable for any nonempty closed sets M and N, initial position (t_0, x_0) and instant $\vartheta \geqslant t_0$.

Proof. We first consider Problem 1 on the encounter with set M inside set N by the instant ϑ. From the halfspace $t \leqslant \vartheta$ we remove those positions (t_*, x_*) for which the following two circumstances obtain simultaneously.

1°. The problem of evading at least one neighborhood $G(M)$ of set M inside at least one neighborhood $H(N)$ of set N up to the instant $\vartheta \geqslant t_*$ is solvable from the position (t_*, x_*) as from the initial position.

2°. An instant t^*, $(t_* < t^* \leqslant \vartheta)$ and a second player's control $v_t^* \in Q_c(x_*; t_*, t^*)$ exist such that under an arbitrary choice of the first player's control $\mu_t^* \in P_c(x_*, t_*, t^*)$ the solution $x(t)$, $t_* \leqslant t \leqslant t^*$ of the differential equation

$$x^{\cdot} = \langle\mu_t^* \otimes v_t^*, f(t, x, u, v)\rangle, \ x(t_*) = x_*$$

satisfies the condition $(t, x, (t)) \notin M$, $t_* \leqslant t \leqslant t^*$.

The set W_u^ϑ of remaining positions possesses the following properties which follow immediately from its construction.

$1_u°$. Set W_u^ϑ is u_c-stable.

$2_u°$ $W_u^\vartheta \subset N$.

$3_u°$. At some instant $\tau \leqslant \vartheta$ the section $W_u^\vartheta(\tau)$ of set W_u^ϑ by the hyperplane Γ_τ is wholly located in the section $M(\tau)$ of set M by the same hyperplane.

Every set satisfying the three conditions listed is called a u_c-stable bridge. Its maximality, and hence, by virtue of Theorem 5.1, its closeness follow from Theorem 7.1 and the method of constructing the bridge W_u^ϑ.

We now consider the problem of evading set $G(M)$ inside set $H(N)$ up to the instant $\vartheta \geqslant t_0$. From the halfspace $t \leqslant \vartheta$ we remove those positions (t_*, x_*) for which the following two circumstances obtain simultaneously.

1°. The problem of encounter with at least one set $\overline{G^*(M)} \subset G(M)$ inside at least one set

$\overline{H^*(N)} \subset H(N)$ by the instant $\vartheta \geqslant t_*$ is solvable from the position (t_*, x_*) as from the initial position.

2°. An instant t^* $(t_* < t^* \leqslant \vartheta)$ and a first player's control $\mu_t{}^* \in P_c(x_*; t_*, t^*)$ exist such that under an arbitrary choice of the second player's control $\nu_t{}^* \in Q_c(x_*; t_*, t^*)$ the solution $x(t)$, $t_* \leqslant t \leqslant t^*$, of the equation

$$x^{\cdot} = \langle \mu_t{}^* \otimes \nu_t{}^*, f(t, x, u, v) \rangle, \ x(t_*) = x_*$$

is such that $(t, x(t)) \in H(N)$ when $t_* \leqslant t \leqslant t^*$.

It can be verified that the set $W_v{}^\vartheta$ of remaining positions is a v_c-stable bridge, i.e., satisfies the conditions:

$W_v{}^\vartheta$ is v_c-stable; $W_v{}^\vartheta \cap G(M) = \varnothing$.

Furthermore, the set $W_v{}^\vartheta$ constructed is a maximal v_c-stable bridge and, consequently, is closed. By the method of construction the sets $W_u{}^\vartheta$ and $W_v{}^\vartheta$ form a partition of the position space $R \times R^n$. The assertion of Theorem 8.1 follows from the fact that sets $W_u{}^\vartheta$ and $W_v{}^\vartheta$ exist and from Theorems 7.1 and 7.2.

In conclusion the author thanks A.I. Subbotin and M.S. Nikol'skii for attention to the work, and also A.A. Agrachev for useful discussions.

REFERENCES

1. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Moscow, NAUKA, 1974.
2. KRASOVSKII N.N. and SUBBOTIN A.I., An alternative for the game problem of convergence. PMM Vol.34, No.6, 1970.
3. ALEKSEICHIK M.I., Subsequent formalization of the fundamental elements of an antogonistic differential game. In: Mathematical Analysis and Its Applications, Vol.7, Rostov, Izd. Rostovsk. Univ., 1975.
4. VAKHRAMEEV S.A., Alternative conditions for the solvability of an encounter-evasion differential game. PMM Vol.45, No.1, 1981.
5. WARGA J., Optimal Control of Differential and Functional Equations. New York, Academic Press, Inc., 1972.
6. CASTAING Ch. Sur les multi — application measurables. — Rev. Française Inf. Rech. Oper. Vol.1, 1967.
7. GAMKRELIDZE R.V., Fundamentals of Optimal Control. Tbilis, Izd. Tbilissk. Univ., 1977.

Translated by N.H.C.